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Green Function on the Two Dimensional Lattice

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Abstract: In quantum field theories the Green functions are useful to extract the properties of interaction between particles. We investigate the Green function for a massless boson defined on a two-dimensional Euclidean lattice, and demonstrate numerical methods to obtain precise values of the lattice Green function. The comparison with the continuum Green function exhibits a characteristic difference at short distances, which is quantified approximately by an exponential function as in the three dimensional case.

Key Words: Green function, Propagator, Lattice formulation

1. Introduction

The Green functions, which we also call the propagators, can be used to extract information on the interaction between particles in quantum field theory. In a D dimensional space, the Green function $G(x)$ at $x = (x_1, x_2, ..., x_D)$ is defined by the relation

$$\Delta G(x) = -\delta^{(D)}(x) \ , \tag{1}$$

where Δ is the Laplacian and $\delta^{(D)}(x)$ the Dirac delta function defined in the same D dimensional space.

A successful example of the use of propagators can be found in quantum electrodynamics in four spacetime dimensions. In this theory, the interaction between electric charges is described by the exchange of massless photons. The photon propagator in the momentum space has the form $1/p^2$ with the momentum p_μ, and its Fourier transform in the three dimensional space leads to the Green function of the form $G(x) = 1/(4\pi|x|)$ for the spatial distance $|x|$, which is nothing but the Coulombic potential.

Another example is for the massive scalar field theory, where the propagator in the momentum space is modified to the form $1/(p^2+m^2)$ due to the mass of particle m, and the corresponding potential is modified to be $G(x) = e^{-m|x|}/(4\pi|x|)$, which indicates that the interaction range is limited to short distances of $|x| < 1/m$ due to the exponential factor $e^{-m|x|}$. This feature is, for instance, applied to describing the properties of short ranged weak interactions.

It should be noted that the common $1/|x|$ behavior in the above examples reflects the fact that the space dimension is just three (cf. the Gauss law). In other words, if the space dimension is not three, the propagator shows a different behavior on $|x|$. In some cases, the physical system has a certain symmetry on the spacetime or acquires such a spacetime symmetry through dynamics, and then, the effective spatial dimension is reduced. Thus, it is interesting to investigate the Green function in lower dimension, which may be useful to clarify not only the properties of the mediated particles, but also the underlying spacetime symmetry in the system.

In this report, we present a part of our study on the Green function in two dimensional space, in particular, defined on a discrete lattice. This study is continuation of our previous works in three dimensions [1, 2]. In the continuum theory, it is known that the Green function contains peculiar divergences both at the origin and at long distances, which make it difficult to obtain precise values of the Green function with a naive numerical manner. We shall investigate how these divergences can be controlled on the lattice with various numerical methods, and compare the lattice result with the continuum one.

2. The lattice Green function

The lattice Green function $G(x)$ for a massless boson in the coordinate space is generally defined by the relation

$$\Delta G(x) = -\delta_{x0} \ , \tag{2}$$

where Δ denotes the lattice Laplacian and δ_{x0} is the Kronecker delta, $\delta_{x0} = 1$ for $x = 0$ and $\delta_{x0} = 0$ for $x \neq 0$. In the two-dimensional Euclidean space $D = 2$, the left-hand side of Eq. (2) is written as

$$\Delta G(x) \ = \ \sum_{\mu=1}^{2} \nabla^*_\mu \nabla_\mu G(x) \ , \tag{3}$$

where ∇_μ and ∇^*_μ are the forward and backward differences to a direction μ, respectively. Note that the lattice spacing is assumed to be one in this report.

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The solution of Eq. (2) is formally obtained by performing the Fourier transformation of the Green function in the momentum space. On the lattice, the momentum carried by a particle is defined by

$$\hat{p}_\mu = 2\sin\frac{p_\mu}{2} \ , \tag{4}$$

where $p_\mu \in [-\pi, \pi]$. The Green function for a massless boson in the momentum space is then given by

$$\tilde{G}(p) = \frac{1}{\hat{p}^2} = \frac{1}{4\sum_\mu \sin^2\frac{p_\mu}{2}} = \frac{1}{4 - 2\sum_\mu \cos p_\mu} \ , \tag{5}$$

and its Fourier transform is formally written as

$$G(x) = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} e^{ipx} \tilde{G}(p) \ , \tag{6}$$

where $p\,x = \sum_{\mu=1}^{2} p_\mu x_\mu$. This is rewritten by using the modified Bessel function of the first kind I_n as

$$G(x) = \int_0^\infty d\alpha\, e^{-4\alpha} \prod_{\mu=1}^{2} I_{x_\mu}(2\alpha) \ , \tag{7}$$

where I_n behaves as

$$I_n(2\alpha) \sim \sqrt{\frac{1}{4\pi\alpha}} e^{2\alpha} \tag{8}$$

for large α. It is clear that the integrand is of $O(\alpha^{-1})$, and hence, the integral will diverge logarithmically. Thus, it is not appropriate to proceed the numerical integration of Eq. (7) before discriminating the divergent part of the integral.

We then use the coordinate space method as demonstrated in Refs. [3–5]. The crucial idea in this method is based on Vohwinkel's observation that the lattice Green function satisfies a relation

$$(\nabla_\mu^* + \nabla_\mu) G(x) = x_\mu \int_{-\pi}^{\pi} \frac{d^Dp}{(2\pi)^D} e^{ipx} \ln(\hat{p}^2) \tag{9}$$

in any D dimensions. Summing over all directions μ in Eq. (9), one obtains a recursion relation,

$$G(x+\hat{\mu}) = G(x-\hat{\mu}) + \frac{2x_\mu}{\rho} \sum_{\nu=1}^{D} [G(x) - G_\infty(x-\hat{\nu})] \tag{10}$$

with $\rho = \sum_{\mu=1}^{D} x_\mu$ for $\rho \neq 0$.

In two dimensions, Eq. (10) is reduced to

$$\begin{aligned} &G(x+\hat{\mu}) = G(x-\hat{\mu}) \\ &+\frac{2x_\mu}{x_1+x_2}(2G(x) - G(x-\hat{1}) - G(x-\hat{2})) \ , \end{aligned} \tag{11}$$

where $G(x)$ is invariant under the permutation of x_1 and x_2 as

$$G(x_1, x_2) = G(x_2, x_1) \ , \tag{12}$$

and the reflection around the origin at $x = 0$ as

$$G(x_1, x_2) = G(-x_1, x_2) = G(x_1, -x_2) \ . \tag{13}$$

By using Eq. (11) and the above symmetries, we immediately obtain

$$G(1,0) = G(0,0) - \frac{1}{4} \ . \tag{14}$$

One may further find that the Green functions at other positions are generally given by the linear combination of $G(0,0)$ and $G(1,1)$, for instance,

$$G(2,0) = 3G(0,0) - 2G(1,1) - 1 \ , \tag{15}$$

$$G(3,0) = 13G(0,0) - 12G(1,1) - \frac{17}{4} \ , \tag{16}$$

$$G(4,0) = \frac{187}{3}G(0,0) - \frac{184}{3}G(1,1) - 20 \ , \tag{17}$$

and so on. We are then left with the task to determine the value of $G(0,0)$ and $G(1,1)$.

For this purpose, we firstly define

$$g_1(n) = G(n,0) \ , \quad g_2(n) = G(n,1) \tag{18}$$

for $n \in \mathbb{Z}$. The use of the Vohwinkel relation then leads to two simultaneous recursion relations

$$g_1(n+1) = 4g_1(n) - g_1(n-1) - 2g_2(n) \ , \tag{19}$$

$$g_2(n+1) = \frac{4ng_2(n)}{n+1} - \frac{(n-1)g_2(n-1)}{n+1} - \frac{2ng_1(n)}{n+1} \ . \tag{20}$$

For these g_1 and g_2, we find an interesting fact that the combination of the form,

$$\begin{aligned} k(n) &= (n-1)g_1(n) + ng_2(n) \\ &\quad -ng_1(n-1) - (n-1)g_2(n-1) \ , \end{aligned} \tag{21}$$

is invariant under the shift of n as

$$\begin{aligned} k(n+1) &= n(4g_1(n) - g_1(n-1) - 2g_2(n)) \\ &\quad +(n+1)(\frac{4n}{n+1}g_2(n) - \frac{n-1}{n+1}g_2(n-1) \\ &\quad -\frac{2n}{n+1}g_1(n)) - (n+1)g_1(n) - ng_2(n) \\ &= 4ng_1(n) - ng_1(n-1) - 2ng_2(n) \\ &\quad +4ng_2(n) - (n-1)g_2(n-1) - 2ng_1(n) \\ &\quad -(n+1)g_1(n) - ng_2(n) \\ &= (n-1)g_1(n) + ng_2(n) \\ &\quad -ng_1(n-1) - (n-1)g_2(n-1) \\ &= k(n) \ , \end{aligned} \tag{22}$$

where we have used Eqs. (19) and (20). Since it is natural to expect that the lattice Green function at long distances behaves like the continuum one [5] such as

$$G(x) = -\frac{1}{4\pi}(\ln x^2 + 2\gamma + 3\ln 2) \tag{23}$$

with the Euler-Mascheroni constant $\gamma = 0.577215...$, the invariant relation for a large n will be

$$\begin{aligned} k(n) \;\to\; & (n-1)(-\frac{1}{4\pi}\ln n^2) + n(-\frac{1}{4\pi}\ln(n^2+1)) \\ & -n(-\frac{1}{4\pi}\ln(n-1)^2) \\ & -(n-1)(-\frac{1}{4\pi}\ln((n-1)^2+1)) \\ \to\; & -\frac{1}{\pi}\,. \end{aligned} \tag{24}$$

Therefore we conclude $k(n) = -1/\pi$ for any value of $n \geq 0$. Inserting $n = 1$ to $k(n)$, we find $k(1) = g_2(1) - g_1(0) = G(1,1) - G(0,0)$. Thus we obtain

$$G(1,1) = G(0,0) - \frac{1}{\pi}\,. \tag{25}$$

It turns out that all Green functions now have a common constant value $G(0,0)$ as

$$G(2,0) \;=\; G(0,0) + \frac{2}{\pi} - 1\,, \tag{26}$$

$$G(3,0) \;=\; G(0,0) + \frac{12}{\pi} - \frac{17}{4}\,, \tag{27}$$

$$G(4,0) \;=\; G(0,0) + \frac{184}{3\pi} - 20\,, \tag{28}$$

and so on. As the lattice Green function at long distances is expected to behave as in Eq. (23), the finite part of the $G(0,0)$ should correspond to $(2\gamma + 3\ln 2)/(4\pi)$. On the other hands, the divergent part has a structure similar to that in the complete elliptic integral of the first kind $K(s)$ with $s \to 1$, which must be renormalized properly.

In principle, the above recursion relations allow us to obtain the Green function at any long distances. However, since the convergence ratio at long distances is not so small due to the logarithmic behavior of the Green function, the accumulation of numerical errors that originates from precision limits may affect the result. For a large n, we have

$$G(n+1,0) - G(n,0) = -\frac{1}{2\pi}\ln\frac{n+1}{n} \simeq -\frac{1}{2\pi n}\,, \tag{29}$$

which is of $O(10^{-4})$ even at $n = 1000$. For comparison the corresponding values are of $O(10^{-7})$ and $O(10^{-10})$ in three and four dimensions, respectively.

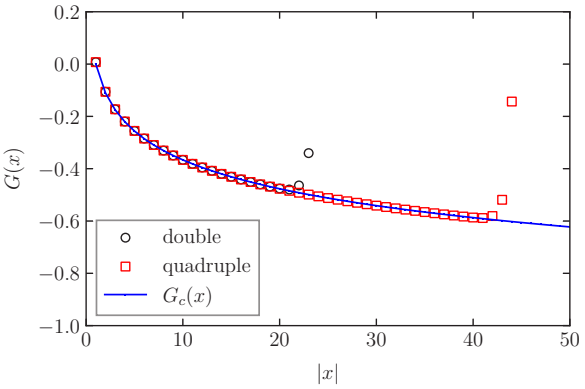

Fig. 1: The behavior of $G(x)$ with the double and quadruple-precision arithmetics, in comparison with the continuum Green function $G_c(x)$.

3. Numerical results

In Fig. 1, we show the Green function along the x_1 axis such as $x = (n, 0)$, which is computed by using Fortran with the double- and quadruple-precision arithmetics, respectively. The lattice results are compared to the continuum Green function

$$G_c(x) = -\frac{1}{4\pi}\ln x^2\,, \tag{30}$$

As noted above, the difficulty arises for controlling numerical errors at around $n = 20$ for the double-precision arithmetic. Even if the quadruple-precision arithmetic is used, available distance is limited to around $n = 40$.

In order to overcome this problem, we propose two alternatives. One is to resort to the arbitrary-precision arithmetic computation, for instance, by using Python with the help of mpmath library [6]. The other is to compute the Green function in a finite volume, but of course, its size should be large enough. As the basic procedure of the former with Python is the same as that with Fortran, we just add some explanations for the latter.

In a finite volume with periodic boundary conditions in all directions, the momentum p_μ in Eq. (4) is discretized as $p_\mu = 2\pi n_\mu / L_\mu$ with integers $n_\mu = 0, 1, 2, ..., L_\mu - 1$, where the lattice volume is assumed to be $L_1 L_2$. The Green function in the momentum space will be

$$\tilde{G}_L(p) = \frac{1}{\hat{p}^2} = \frac{1}{4\sum_\mu \sin^2\frac{p_\mu}{2}} = \frac{1}{4 - 2\sum_\mu \cos\frac{2\pi n_\mu}{L_\mu}}\,, \tag{31}$$

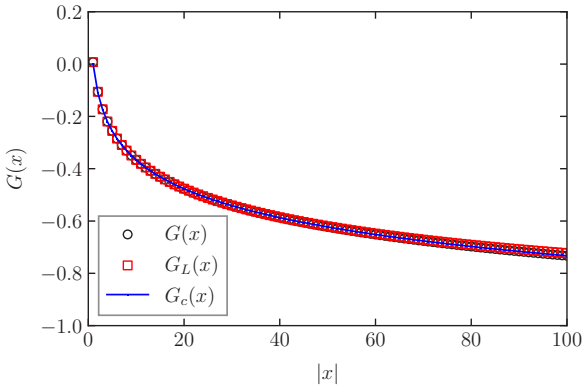

Fig. 2: The behavior of $G(x)$ with the arbitrary-precision arithmetic (keep 100 digits of precision) and of $G_L(x)$ (on 512^2) with the double-precision arithmetic.

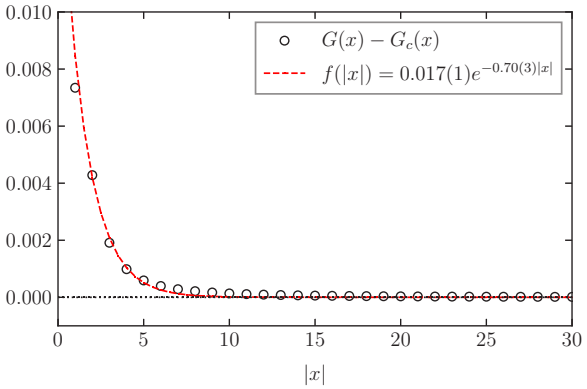

Fig. 3: The difference of the lattice Green function and the continuum one $G(x) - G_c(x)$.

and the Fourier transform is

$$G_L(x) = \frac{1}{L_1 L_2} \sum_{n_\mu \neq 0} e^{i \frac{2\pi n_\mu}{L_\mu} x_\mu} \tilde{G}_L(p) \ , \qquad (32)$$

which can be computed by using the FFT algorithm.

In Fig. 2, we show the two results, where the first calculation is performed by keeping 100 digits of precision during the recurrence, while the second is done with the lattice size of $L_1 \times L_2 = 512^2$. In both cases, computations are achieved within negligible time. Although the computation for the finite volume Green function is done by using Python with the SciPy library, we just stick to the normal double-precision arithmetic. We find that both methods work quite well. The available distance can easily be increased if needed.

As far as Fig. 2 is looked at, it seems that the lattice Green function completely coincides with the continuum one. However, Fig. 3 indicates that this is not the case especially at short distances of $x \leq 7$, where the difference between the lattice Green function and the continuum one, $G(x) - G_c(x)$, is plotted. A chi-square fitting analysis for the difference indicates that it approximately obeys an exponential function

$$f(|x|) = 0.017(1) e^{-0.70(3)|x|} \ . \qquad (33)$$

Note that this tendency is quite the same as in three dimensions [1, 2]. If we take the resulting fitting parameter seriously, the absolute value of the coefficient of $|x|$ in the exponential function 0.70(3) is slightly smaller than that in three dimension 0.84(2) [1], suggesting that the lattice cutoff effect tends to be enhanced in lower dimensions.

4. Summary

We have investigated the lattice Green functions for a massless boson in two-dimensional Euclidean space. We have demonstrated several numerical methods to obtain precise values of the lattice Green function. The comparison with the continuum Green function indicates that the lattice results at short distances contain a characteristic difference from the continuum one as quantified by an exponential function, which seems to have the same structure as in the three-dimensional space [1, 2].

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