The DGL flux-tube solution in the London limit

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Abstract: The dual lattice formulation of the dual Ginzburg-Landau (DGL) theory enables us to obtain the finite-length flux-tube solution for the quark-antiquark system in quantum chromodynamics (QCD) systematically. By using this formulation, we investigate the energy and the field profile of the color-electric flux tube in the London limit, as an extreme case of the dual superconducting QCD vacuum.

Key Words: QCD, Dual superconductor, Flux tube, London limit

1. Introduction

The mechanism of quark confinement in the vacuum of quantum chromodynamics (QCD) is relevant to hadron structures, which is reasonably explained by regarding the QCD vacuum as a dual superconductor [1–3]. The color-electric field between a quark and an antiquark is then squeezed into a flux tube via the dual Meissner effect, leading to a linearly rising potential. This feature is observed by Monte-Carlo simulations of the lattice QCD, and is well captured by the dual Ginzburg-Landau (DGL) theory [4–7].

The quantitative property of the vacuum depends on the type of dual superconducting phase, as in ordinary superconductors, where the type is characterized by the Ginzburg-Landau parameter κ, defined by the ratio of the penetration depth to the coherence length. So far, many efforts have been made to identify the value of κ by investigating the field profile of the quark-antiquark system with the lattice QCD simulations, but it seems that the results are not easily settled. For the determination, a careful analysis of the lattice QCD results is required, with the quantitative results of the color-electric flux-tube solution in the DGL theory.

Recently, we have completed a robust numerical method for solving field equations of the DGL theory [8], and demonstrated that any finite-length flux-tube solutions can be obtained systematically. The essential aspect of our method is to formulate the DGL theory on the dual lattice. In the DGL theory, the open Dirac string is necessarily introduced to define a quark and an antiquark at both endpoints of the string. However, the Dirac string, described by the Dirac delta function in the continuum theory, is not suitable for numerical treatment. On the dual lattice, this problem is successfully avoided, since the Dirac string is regularized as integer plaquette variables. We thus obtain regularized field equations to be solved numerically.

In this report, using the dual lattice formation of the DGL theory, we examine the flux-tube solution in the type-II limit, $\kappa \to \infty$, also referred to as the London limit. This limit has a practical importance in ordinary superconductors, as the type-II superconductors with larger κ can remain superconducting in much higher magnetic fields [9], making them useful for industrial applications. Although we intend to apply the flux-tube solution for the analysis of lattice QCD results, where the value of κ may be finite, it will be useful to know the properties of the flux tube in such an extreme case in advance. Note that our method can also apply to the ordinary superconductor by regarding the dual variables as the original ones.

2. Numerical procedure

We consider Euclidean space in D dimensions, whose coordinate is specified by $x = (x_1, x_2, ..., x_D)$. The DGL theory is defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{4}{}^*F_{\mu\nu}^2 + |(\partial_\mu + igB_\mu)\chi|^2 + \lambda(|\chi|^2 - v^2)^2 , \quad (1)$$

where B_μ denotes the axial vector dual gauge field and χ the complex scalar monopole field, and

$$ {}^*F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - e\Sigma_{\mu\nu} \quad (2)$$

is the dual field strength tensor (the star represents the Hodge dual). The Greek indices μ and ν take values from 1 to D, and repeated indices are summed over. The monopole field carries the magnetic charge g, while the quark and antiquark fields, which are introduced as the endpoints of the Dirac string $\Sigma_{\mu\nu}$, carry the electric charge e and $-e$. The magnetic and electric charges satisfy the Dirac quantization condition, $eg = 2\pi$. The U(1) dual gauge symmetry, the invariance of the Lagrangian density with the transformation of the field variables as $\chi \to \chi e^{i\xi}$, $\chi^* \to \chi^* e^{-i\xi}$, $B_\mu \to B_\mu - \partial_\mu \xi$ for $\xi \in \Re$, where the star denotes the complex conjugate, will be broken spontaneously once monopole condensation occurs $|\chi| \to v$ (vacuum expectation value). Then, both the dual gauge field and

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the monopole field acquire masses, $m_B = \sqrt{2}gv$ and $m_\chi = 2\sqrt{\lambda}v$, respectively. The ratio of these masses,

$$\kappa = \frac{m_\chi}{m_B} = \frac{\sqrt{2\lambda}}{g} , \tag{3}$$

characterizes the type of dual superconducting phase; type-I for $\kappa < 1$ and type-II for $\kappa > 1$. We may call κ the Ginzburg-Landau (GL) parameter as in the ordinary superconductor. Note that the inverses of these masses, m_B^{-1} and m_χ^{-1}, correspond to the penetration depth and the coherence length.

In the dual lattice formulation, we firstly isolate the physical scale v from the Lagrangian density by introducing dimensionless variables with the carets,

$$\begin{aligned}
&gB_\mu \equiv v\hat{B}_\mu \ , \quad \chi \equiv v\hat{\phi}e^{i\eta} \ , \\
&\partial_\mu \equiv v\hat{\partial}_\mu \ , \quad x_\mu \equiv v^{-1}\hat{x}_\mu \ , \quad \Sigma_{\mu\nu} \equiv v^2\hat{\Sigma}_{\mu\nu} \ , \\
&m_B \equiv v\hat{m}_B \ , \quad m_\chi \equiv v\hat{m}_\chi \ ,
\end{aligned} \tag{4}$$

and secondary, define the monopole field as the *site* variables, the dual gauge field as the *link* variables. We assume that the lattice volume is given by $L^D \equiv \prod_{\mu=1}^{D} L_\mu$ ($L_\mu \in \mathbb{N}$), and impose periodic boundary conditions in all directions, as in the ordinary lattice QCD simulations. Finally, by summing over all sites $x = (x_1, x_2, ..., x_D)$ with $x_\mu \in [1, L_\mu]$, we obtain the DGL action on the dual lattice,

$$\begin{aligned}
S = \beta_g \sum_x \Bigg[&\frac{1}{4} {}^*F_{\mu\nu}(x)^2 \\
&+\hat{m}_B^2 \sum_{\mu=1}^{D} \Big(\phi(x)^2 - \phi(x)\phi(x+\hat{\mu}) \cos B_\mu(x) \Big) \\
&+ \frac{\hat{m}_B^2 \hat{m}_\chi^2}{8} \left(\phi(x)^2 - 1\right)^2 \Bigg] ,
\end{aligned} \tag{5}$$

where we omit all of the caret for simplicity *except for the masses*, and the dual gauge field is redefined as $B_\mu + \partial_\mu \eta(x) \to B_\mu(x)$. $\hat{\mu}$ is assumed to be a unit vector to the direction μ. The dual field strength in Eq. (2) is defined as the non-compact *plaquette* variables,

$${}^*F_{\mu\nu}(x) = (\partial \wedge B)_{\mu\nu}(x) - 2\pi\Sigma_{\mu\nu}(x) , \tag{6}$$

where $(\partial \wedge B)_{\mu\nu}(x) \equiv B_\mu(x) + B_\nu(x+\hat{\mu}) - B_\mu(x+\hat{\nu}) - B_\nu(x)$. The overall factor $\beta_g \equiv 1/g^2$ in Eq. (5) is the dual gauge coupling. In the classical treatment, β_g does not affect the dynamics of the solution. Therefore, we assume $\beta_g = 1$ when numerical results are presented in the following.

According to Eq. (4), the lattice spacing a can be defined by $a \equiv 1/v = \hat{m}_B/m_B$. This means that we can always compare the distances for various $\hat{m}_B$ values with each other in units of $1/m_B$, the penetration depth. The small $\hat{m}_B$ or large $\hat{m}_B$ corresponds to the lattice with the fine or coarse discretization.

For a large κ, the monopole field prefers to be $\phi = 1$ everywhere to reduce the energy from the monopole self-interaction term. On the other hand, if this occurs, the energy from the monopole kinetic term will diverge as this term contains interaction between the Dirac string and the monopole field via the singular part of the dual gauge field. However, this is the problem within the continuum theory. On the dual lattice, such a potential ultraviolet divergence from the Dirac string is already regularized by admitting a finite size of $\Sigma_{\mu\nu}$, so that the monopole field can take nonzero values on all sites of the lattice. This feature allows us to investigate the type-II limit of the DGL theory at any lattice spacing.

By setting $\phi(x) = 1$ for all sites x in Eq. (5), we obtain the DGL action in the type-II limit,

$$S = \beta_g \sum_x \left[\frac{1}{4} {}^*F_{\mu\nu}(x)^2 + \hat{m}_B^2 \sum_{\mu=1}^{D} (1 - \cos B_\mu(x)) \right] . \tag{7}$$

The field equation for the dual gauge field is derived by

$$\frac{\partial S}{\partial B_\mu(x)} = \beta_g X_\mu(x) = 0 , \tag{8}$$

where

$$X_\mu(x) = \sum_{\nu \neq \mu} [F_{\mu\nu}(x) + F_{\nu\mu}(x-\hat{\nu})] + \hat{m}_B^2 \sin B_\mu(x) . \tag{9}$$

The field equation $X_\mu(x) = 0$ is nothing but the lattice version of the London equation. To solve this field equation, the Newton-Raphson method is applicable; using the derivative of the field equation with respect to B_μ,

$$\delta_B X_\mu(x) \equiv \frac{\partial X_\mu(x)}{\partial B_\mu(x)} = 2(D-1) + \hat{m}_B^2 \cos B_\mu(x) , \tag{10}$$

the dual gauge field is updated iteratively by

$$B_\mu(x) \to B_\mu^{\mathrm{new}}(x) = B_\mu(x) - \frac{X_\mu(x)}{\delta_B X_\mu(x)} . \tag{11}$$

The iteration process is terminated when $\max(|X_\mu(x)|) < \epsilon$ ($\mu = 1, 2, ..., D$) is satisfied for all sites x, with a reasonably small value of ϵ.

To reveal the structure of the solution, the Hodge decomposition can be applied to the dual gauge field [8] as $B_\mu = B_\mu^{\rm reg} + B_\mu^{\rm sing}$ by using the massless lattice Green function $G_L(x)$ in a finite volume L^D, which satisfies the relation

$$\Delta_L G_L(x-x') = \partial_\mu \partial'_\mu G_L(x-x') = -\delta_{xx'} + \frac{1}{L^D} \ , \quad (12)$$

where ∂_μ and ∂'_μ are forward and backward differences, respectively. The decomposed parts of the dual gauge field are then given by

$$B_\mu^{\rm reg}(x) \ = \ \sum_{x'} G_L(x-x')\partial'_\nu {}^*F_{\mu\nu}(x') \ , \quad (13)$$

$$B_\mu^{\rm sing}(x) \ = \ 2\pi \sum_{x'} G_L(x-x')\partial'_\nu \Sigma_{\mu\nu}(x') \ . \quad (14)$$

For a reasonably large lattice volume, the $1/L^D$ term in Eq. (12) can be neglected, and the rotation of $B_\mu^{\rm reg}$ and $B_\mu^{\rm sing}$ become

$$(\partial \wedge B^{\rm reg})_{\mu\nu}(x) \ = \ {}^*F_{\mu\nu}(x) - 2\pi C_{\mu\nu}(x) \ , \quad (15)$$

$$(\partial \wedge B^{\rm sing})_{\mu\nu}(x) \ = \ 2\pi\Sigma_{\mu\nu}(x) + 2\pi C_{\mu\nu}(x) \ , \quad (16)$$

where $C_{\mu\nu}(x) = -\epsilon_{\mu\nu\sigma}\partial'_\sigma \sum_{x'} G_L(x-x') j_0(x')$ represents the Coulombic electric field. The dual field strength becomes

$${}^*F_{\mu\nu}(x) = (\partial \wedge B^{\rm reg})_{\mu\nu}(x) + 2\pi C_{\mu\nu}(x) \ . \quad (17)$$

Since the contribution from the cross term of $(\partial \wedge B^{\rm reg})_{\mu\nu}$ and $C_{\mu\nu}$ in Eq. (17) vanishes because of the relation $\epsilon_{\lambda\mu\nu}\partial_\lambda(\partial \wedge B^{\rm reg})_{\mu\nu} = 0$, the action can be decomposed into two parts as $S = S_{\rm coul} + S_{\rm sole}$, where

$$S_{\rm coul} \ \equiv \ \beta_g \pi^2 \sum_x C_{\mu\nu}(x)^2 \ , \quad (18)$$

$$S_{\rm sole} \ \equiv \ \beta_g \sum_x \left[\frac{1}{4}[(\partial \wedge B^{\rm reg})_{\mu\nu}(x)]^2 + \hat{m}_B^2 \sum_{\mu=1}^{D} (1-\cos B_\mu(x)) \right] \ . \quad (19)$$

The labels "coul" and "sole" mean the Coulombic and the solenoidal parts, respectively, reflecting the structure of the resulting electric field profile. Note that $S_{\rm coul}$ vanishes in $D=2$ dimensions, and $S_{\rm coul}$ becomes the Coulombic potential in $D=3$ dimensions.

3. Numerical results

For a quark-antiquark system with the charges N_q and $-N_q$, where the quark and antiquark are separated at a distance R along the x_3-axis in $D=3$ dimensions (in the computation, $N_q = 1$ is set), we set a connected stack of N_q as

$$\Sigma_{12}(0,0,l) = -N_q \quad (l = 1, 2, ..., R) \ , \quad (20)$$

and set zero for the rest. The total electric charge density in the system is given by $j_0(x) = N_q \delta_{x_1 0}\delta_{x_2 0}(\delta_{x_3 0} - \delta_{x_3 R})$.

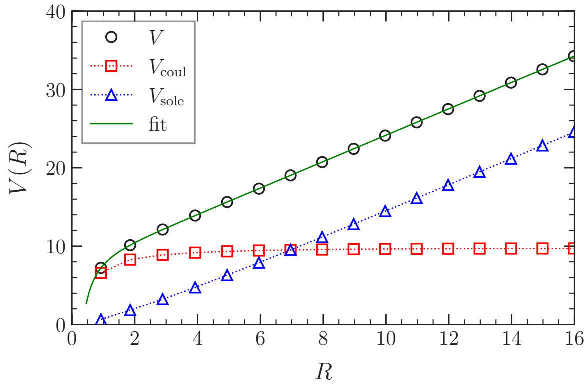

Fig. 1: The interquark potential with the Hodge decomposition $V = V_{\rm coul} + V_{\rm sole}$ for $\hat{m}_B = 0.50$ on the $L^3 = 32^3$ lattice, where the tree-level improvement [10] is applied to the distance R. The solid curve on V is the fitting result.

In Fig. 1, we present the potential energy V (the converged value of S) with the Hodge decomposition, $V_{\rm coul}$ and $V_{\rm sole}$, as a function of the distance R for $\hat{m}_B = 0.50$ on the $L^3 = 32^3$ lattice, where the tree-level improvement [10] is applied to the distance R, by using massless lattice Green functions in an infinite volume. The iteration process is terminated when the maximum violation becomes smaller than $\epsilon = 10^{-6}$.

The χ^2 fitting analysis suggests that the functional form of V is well described by

$$V_{\rm fit}(R) = -\frac{4.0(1)e^{-1.05(3)R}}{R} + 1.6847(5)R + 7.310(6) \ , \quad (21)$$

where the slope of the linear term, originating from $V_{\rm sole}(R)$, is consistent with our empirical finding of the string tension computed in the $D=2$ setting [8],

$$\sigma/m_B^2 = \frac{\beta_g \pi |N_q|}{2} \ln(1 + \frac{18.0(1)}{\hat{m}_B^2}) \ . \quad (22)$$

$V_{\rm coul}(R)$ exactly coincides with the lattice Coulombic potential of the functional form

$$V_{\rm coul}(R) = 4\pi^2 \beta_g N_q^2 [G_L(0,0,0) - G_L(0,0,R)] \ , \quad (23)$$

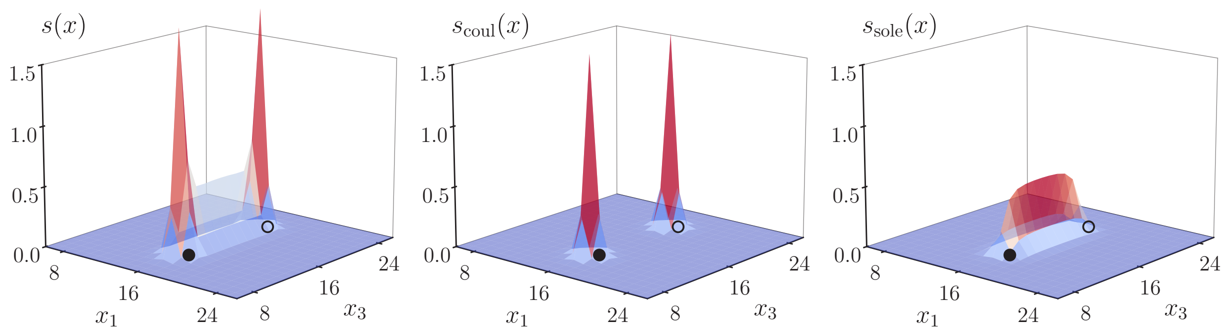

Fig. 2: The profile of the action density s (left) with the Hodge decomposition, the Coulombic part $s_{\rm coul}$ (middle), and the solenoidal part $s_{\rm sole}$ (right), just on the median plane that the quark (the black-filled circle) and antiquark (the black-open circle) are placed for $\hat{m}_B = 0.50$ on the $L^3 = 32^3$ lattice, where the interquark distance is $R = 10$.

where G_L is the massless lattice Green funtion on the $L^3 = 32^3$ lattice. The Yukawa term in $V_{\rm fit}$ indicates that $V_{\rm coul}(R)$ is modified by the nontrivial short-distance behavior in $V_{\rm sole}$. An interesting observation may be that the effective mass in the Yukawa term, in units of m_B, is $m/a = 1.05(3)/a = 2.10(6)\,[{\rm m_B}]$, which is larger than one. This suggests that the Yukawa term in the type-II limit decays more quickly than previously thought $\sim e^{-m_B r}/r$ for $r = Ra$ [1, 6].

The interquark potential has a direct connection to the profile of the action density $s(x)$, as $V = \sum_x s(x)$ in $D = 3$ dimensions. We present the profile in Fig. 2 for the $R = 10$ case just on the median plane of the flux tube. The action density $s(x)$ can be decomposed into two parts $s_{\rm coul}(x)$ and $s_{\rm sole}(x)$ as in the potential. We observe two sharp peaks at the location of the quark and antiquark in the Coulombic part, while a ridge structure between the quark and antiquark in the solenoidal part.

The height of the two sharp peaks themselves is irrelevant to the interaction, as it reflects the self-energies of the quark and antiquark. The interaction occurs when positive values continuously connect the action density. When the quark and antiquark are close to each other, the tails of the two shape peaks overlap, yielding the Coulombic potential. As the quark and antiquark are separated, the overlap of the tails is reduced, and the ridge structure becomes dominant. As the ridge structure persists with keeping the height and width even when the quark and antiquark are separated further, we obtain the linearly-rising potential.

4. Summary

The dual lattice formulation of the DGL theory enables us to obtain the finite-length flux-tube solution for the quark-antiquark system in QCD systematically. An essential aspect of the dual lattice formulation is that the Dirac string singularity in the continuum DGL theory, which is necessarily introduced to define the color-electric charges in the dual description, is regularized to suit numerical treatment. By using this formulation, we have examined the energy and the field profile of the color-electric flux tube in the London limit. The further investigation will be useful to expose the properties of the flux tube in this extreme limit.

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