The Dual Abelian Higgs Model in Two-dimensional Space

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Abstract: The dual Abelian Higgs (DAH) model is one of the nonperturbative effective models of quantum chromodynamics for strong interaction inside hadrons. The DAH model possesses a classical string-like flux-tube solution, which can be applied to explaining the quark confinement mechanism. We study the flux-tube solution in the DAH model in two-dimensional space by solving the field equations numerically, and present some of selected results of the solution.

Key Words: Strong interaction, Quark confinement, Superconductor, Flux tube

I. INTRODUCTION

Quarks and gluons are ingredients of hadrons, and their dynamics is believed to be described by quantum chromodynamics (QCD), a quantum gauge field theory of SU(3) gauge symmetry. However, the investigation of QCD requires nonperturbative methods due to strong nature of the interaction. One of the nonperturbative methods is to change QCD to a weakly interacting theory by applying duality transformation. The dual Abelian Higgs (DAH) model [1, 2], which we deal with in this report, is constructed in such a way [3]. It is then possible to tackle some of the nonperturbative properties of QCD based on classical solutions of the DAH model.

The Lagrangian density of the DAH model is given by

$$\mathcal{L} = \frac{1}{4}{}^*F_{\mu\nu}^2 + |(\partial_\mu + igB_\mu)\chi|^2 + \lambda(|\chi|^2 - v^2)^2 \ , \quad (1)$$

where $B_\mu(x)$ and $\chi(x)$ are the axial-vector dual gauge field and the complex-scalar Higgs field, respectively, and

$${}^*F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - \frac{e}{2}\Sigma_{\mu\nu} \quad (2)$$

is the dual field strength tensor. The model is invariant under the U(1) dual gauge transformation, $B_\mu \to B_\mu - \partial_\mu\xi$, $\chi \to \chi e^{i\xi}$, and $\chi^* \to \chi^* e^{-i\xi}$. The strength of the interaction is controlled by the dual gauge coupling g and the Higgs self coupling λ, where g satisfies the Dirac quantization condition $(e/2)g = 2\pi$ with the color-electric charge $e/2$. An important feature is that the strong coupling with respect to e is now changed to the weak coupling g. The Higgs condensate v is related to the physical scale of the DAH model. The theoretical structure of the DAH model is quite the same as the Ginzburg-Landau (GL) model for describing ordinary superconductor except for the presence of the Dirac string $\Sigma_{\mu\nu}$. In the DAH model, $\Sigma_{\mu\nu}$ leads to a string-like flux-tube solution corresponding to the Abrikosov vortex in the GL model [4].

In this report, we present some of selected results of the flux-tube solution in the DAH model in two-dimensional space obtained by solving the field equation numerically.

II. THE NUMERICAL PROCEDURES

For the numerical method it is useful to isolate the physical scale from the model. In practice, we may introduce dimensionless variables with carets by

$$gB_\mu \equiv v\hat{B}_\mu \ , \quad \chi \equiv v\hat{\chi} = v(\hat{\phi}_1 + i\hat{\phi}_2) \ ,$$
$$\partial_\mu \equiv v\hat{\partial}_\mu \ , \quad x_\mu \equiv v^{-1}\hat{x}_\mu \ , \quad \Sigma_{\mu\nu} \equiv v^2\hat{\Sigma}_{\mu\nu} \ . \quad (3)$$

The action of the DAH model in two-dimensional space is then written as

$$S = \beta v^2 \int d^2\hat{x}\left[\frac{1}{4}{}^*\hat{F}_{\mu\nu}^2 + \frac{\hat{m}_B^2}{2}(\hat{D}_\mu\hat{\phi}_a)^2 + \frac{\hat{m}_B^2\hat{m}_\chi^2}{8}(\hat{\phi}_a^2 - 1)^2\right], \quad (4)$$

where

$${}^*\hat{F}_{\mu\nu} = \hat{\partial}_\mu\hat{B}_\nu - \hat{\partial}_\nu\hat{B}_\mu - 2\pi\hat{\Sigma}_{\mu\nu} \ , \quad (5)$$
$$\hat{D}_\mu\hat{\phi}_a = \hat{\partial}_\mu\hat{\phi}_a - \epsilon_{ab}\hat{B}_\mu\hat{\phi}_b \ . \quad (6)$$

ϵ_{ab} in the covariant derivative in Eq. (6) is the 2nd-rank antisymmetric tensor ($\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$). The repeated Greek and Latin indices are to be summed over from one to two. Three parameters g, λ, and v in Eq. (1) are now translated to the inverse square of the dual gauge coupling $\beta = 1/g^2$, and the masses of the dual gauge field and the Higgs field,

$$m_B = \sqrt{2}gv \equiv \hat{m}_B v \ , \quad m_\chi = 2\sqrt{\lambda}v \equiv \hat{m}_\chi v \ . \quad (7)$$

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We further formulate the DAH model in Eq. (4) on the dual lattice. We consider the lattice volume of the size $L^2 \equiv L_x L_y$, and set the lattice spacing $a = v^{-1}$, where periodic boundary conditions are imposed in both x and y directions. We may set $a = 1$ and recover the physical scale whenever needed. Then, the action of the DAH model has the form

$$\sigma = \sum_{\vec{x}} s(\vec{x}) \tag{8}$$

with

$$s(\vec{x}) = \beta \Bigg[\frac{1}{4} {}^*F_{\mu\nu}(\vec{x})^2 + \frac{m_B^2}{2} (D_\mu \phi_i(\vec{x}))^2 + \frac{m_B^2 m_\chi^2}{8} \left(\phi_i(\vec{x})^2 - 1\right)^2 \Bigg] , \tag{9}$$

where we have omitted all carets for simplicity. Although we use argument $\vec{x}$ to express space coordinates as in a continuum theory, it should be regarded as discretized site variables. On the lattice, we may write the dual field strength tensor as the noncompact plaquette variables,

$$ {}^*F_{\mu\nu}(\vec{x}) = B_\mu(\vec{x}) + B_\nu(\vec{x}+\hat{\mu}) - B_\mu(\vec{x}+\hat{\nu}) - B_\nu(\vec{x}) - 2\pi \Sigma_{\mu\nu}(\vec{x}) . \tag{10}$$

The lattice version of the covariant derivative becomes

$$D_\mu \phi_i(\vec{x}) = \phi_i(\vec{x}) - \phi_i(\vec{x}+\hat{\mu}) \cos B_\mu(\vec{x}) + \epsilon_{ij} \phi_j(\vec{x}+\hat{\mu}) \sin B_\mu(\vec{x}) . \tag{11}$$

If we put a single Dirac string on the two-dimensional plane, we will obtain a flux-tube solution. In this case, σ corresponds to the energy of the flux tube per unit length, which we may call the string tension. $s(\vec{x})$ is then regarded as the energy density at an arbitrary location of $\vec{x} = (x, y)$.

The field equations for the dual gauge field $B_\mu(\vec{x})$ and the Higgs field $\phi_i(\vec{x})$ are given by

$$\frac{\partial V}{\partial B_\mu(\vec{x})} = \beta m_B^2 X_\mu(\vec{x}) = 0 \quad (\mu = 1, 2) , \tag{12}$$

$$\frac{\partial V}{\partial \phi_i(\vec{x})} = \beta m_B^2 Y_i(\vec{x}) = 0 \quad (i = 1, 2) . \tag{13}$$

We solve these field equations simultaneously so as to satisfy $X_\mu(\vec{x}) = 0$ and $Y_i(\vec{x}) = 0$. We use the Newton-Raphson method for this purpose. An advantage of the dual lattice formulation is that it is quite easy to investigate the flux-tube system; one just sets nonzero *integer* values for $\Sigma_{\mu\nu}$ at any desired locations.

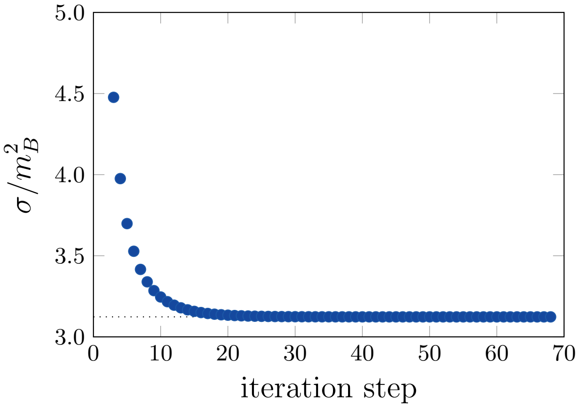

FIG. 1: History of the string tension σ as a function of iteration step for $\beta = 1$ and $m_B = m_\chi = 0.50$ with $N_q = 1$. The dotted line corresponds to the convergence value of $\sigma = 3.1232...$ for $\epsilon = 10^{-4}$.

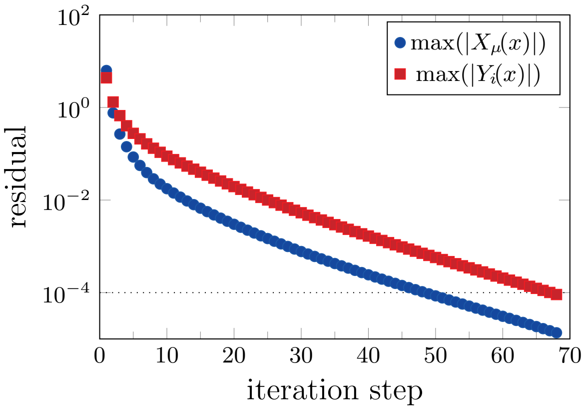

FIG. 2: Histories of the maximum residual of the field equations max($|X_\mu(x)|$) and max($|Y_i(x)|$) as a function of iteration step for $\beta = 1$ and $m_B = m_\chi = 0.50$ with $N_q = 1$. The dotted line is the required level of the residual.

III. THE NUMERICAL RESULTS

As a demonstration of our numerical method, let us put a single Dirac string with the quark charge $N_q = 1$ at the center of lattice $\vec{x} = (\frac{L_x}{2}, \frac{L_y}{2})$, and solve the field equations with the parameters $\beta = 1$ and $m_B = m_\chi = 0.50$ on the lattice size $L^2 = 32^2$. The so-called GL parameter is given by $\kappa \equiv m_B/m_\chi = 1$, which corresponds to just the border of the type-I and type-II superconducting phases [5].

In Figs. 1 and 2, we first show the string tension

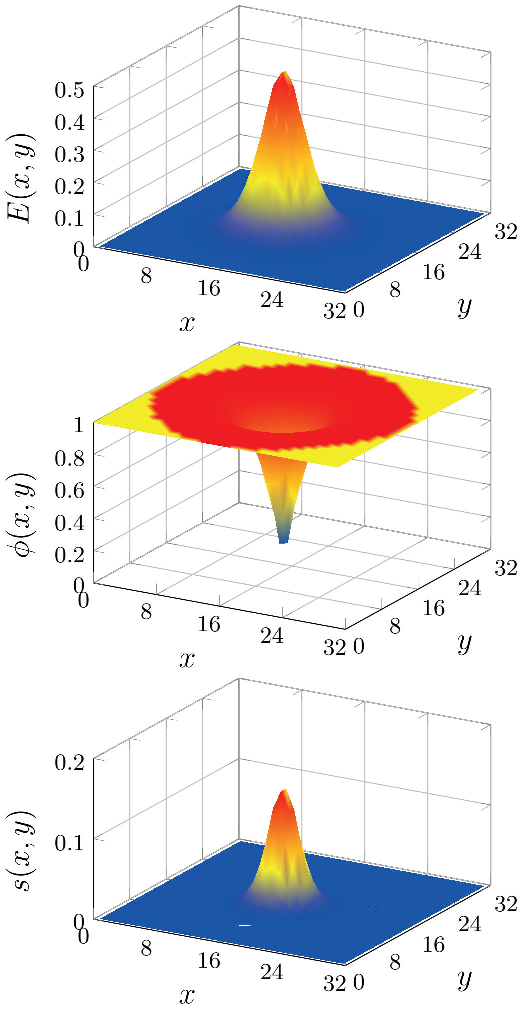

FIG. 3: The profiles of the electric field $E = {}^*F_{12}$ (upper), the Higgs field $\phi = \sqrt{\phi_1^2+\phi_2^2}$ (middle), and the action density s (lower).

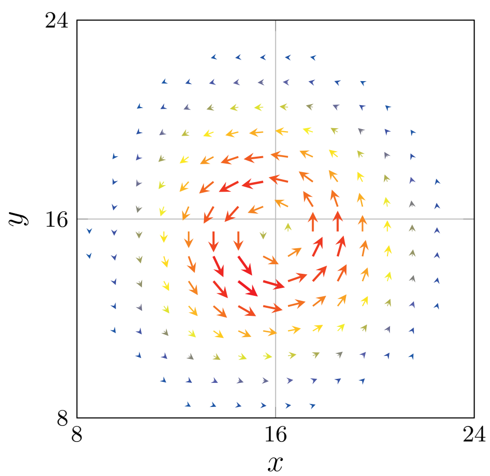

FIG. 4: The profile of the supercurrent k_μ

and the residual of the field equation as a function of the iteration step of the Newton-Raphson method. In this computation, we have set the convergence criterion of the field equations to be $|X_\mu| < \epsilon$ and $|Y_i| < \epsilon$ with $\epsilon = 10^{-4}$. We find that although both the maximum residuals of the field equations gradually reduce to the desired level of ϵ, the string tension quickly converges with the final value. Note that we look at these quantities whenever the new mass parameters and the lattice volume are set. Our examination exhibits that $\epsilon = 10^{-4}$ is already small enough to regard the obtained result as the final solution.

It is possible to compute the analytical continuum value of the string tension at $\kappa = 1$ [5]; it becomes $\sigma/(\beta N_q m_B^2) = \pi$. The relative error of the numerical value of the string tension is evaluated as

$$\frac{\Delta\sigma}{\sigma} = \frac{|\pi - 3.1232...|}{\pi} \simeq 0.0058 \ . \qquad (14)$$

This value can further be smaller when the large volume limit and the continuum limit are taken into account.

In Fig. 3, we then plot the profile of the electric field, the Higgs field, and the action density. In Fig. 4, we also plot the profile of the supercurrent

$$\begin{aligned} k_\mu(x) \;&=\; -m_B^2\{\phi_i(x)\phi_i(x+\hat\mu)\sin B_\mu(x) \\ &\quad +\epsilon_{ij}\phi_i(x)\phi_j(x+\hat\mu)\cos B_\mu(x)\} \ . \end{aligned} \qquad (15)$$

It is clear that the peak structure of the electric field and the action density as well as the vortex structure of the supercurrent exhibit the presence of the flux tube. Note that the obtained field profiles coincide with that of the cylindrical solution as presented in our previous report [6].

The advantage of the present method is that we can examine various type of *multi* flux-tube systems by putting the *multi* Dirac strings on a two dimensional plane. The iteration process naturally finds the energy minimum solution for the given total number of N_q. Let us then demonstrate the usefulness of the method by investigating the two-body flux-tube system. We may put two Dirac strings with the quark charge $N_q = 1$ at $((L_x - r)/2, L_y/2)$ and $((L_x + r)/2, L_y/2)$, so that the original distance between two flux tubes is set to be $r = 6$. We examine three types of the

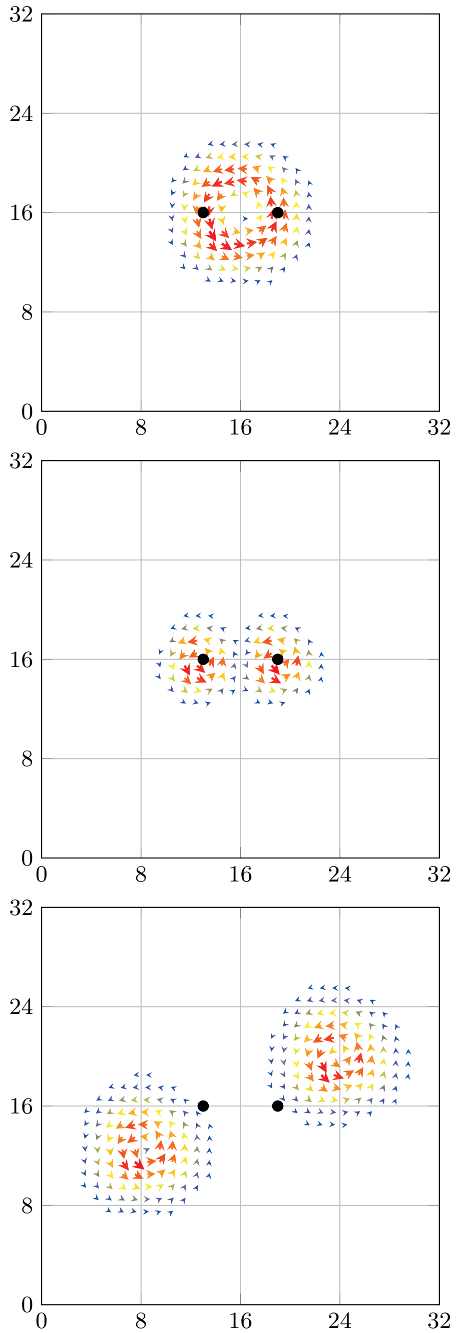

FIG. 5: The profiles of the supercurrent k_μ for $\kappa = 0.5$ (upper), $\kappa = 1$ (middle), and $\kappa = 2$ (lower). The two black points on the xy plane are the original location of the Dirac strings.

superconducting phases $\kappa = 0.5$ (type-I), 1 (border), and 2 (type-II).

In Fig. 5, we summarize results of the supercurrent. The final location of the center of the vortex indicates the nature of the flux-tube interaction, such that the interaction is attractive for type-I, and is repulsive for type-II. The final distance between two flux tubes for the type-II phase depends on the explicit value of κ, and also the boundary condition of the two-dimensional plane. At the border of the two phases there seems to be no interaction although the two vortices are overlapped. In fact, this feature is expected from the property of the string tension of a single cylindrical flux-tube solution with higher charges, where it is just proportional to the charge N_q at the border of the two phases, while it takes smaller (larger) value for type-I (type-II).

We finally note that if we put two Dirac strings with $N_q = 1$ and $N_q = -1$, the two flux tubes immediately disappear by cancellation of the charges regardless of the type of superconducting phase. This means that the interaction is always attractive for the flux tube and *anti*-flux tube system.

IV. SUMMARY

We have investigated the flux-tube solution in the DAH model in two-dimensional space by using the numerical method. We have demonstrated that not only the single flux-tube system but also the multi flux-tube system can be solved by the present method, which allows us to perform further quantitative studies. These results will also be useful to examine the systematic effect for general flux-tube systems in three-dimensional space [7].

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