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Anatomy of the flux-tube solution in the U(1) dual Abelian Higgs model

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The dual Abelian Higgs (DAH) model is one of the nonperturbative effective models of quantum chromodynamics for strong interaction. The DAH model possesses a classical string-like flux-tube solution, which can be applied to explaining the quark confinement mechanism. We discuss analytical properties of the flux-tube solution in the DAH model, and demonstrate a numerical method for solving the field equations.

Key Words: Strong interaction, Quark confinement, Flux tube

I. INTRODUCTION

The investigation of quantum chromodynamics (QCD) for quarks and gluons requires nonperturbative techniques due to strong nature of the interaction. One of such techniques is to change the theory to a weakly interacting one by duality transformation. The dual Abelian Higgs (DAH) model [1, 2], which we handle in this report, is constructed in such a way [3]. It then allows us to investigate some of the nonperturbative properties of QCD based on the classical solution of the model.

The DAH model consists of the axial-vector dual gauge field $B_\mu(x)$ and the complex-scalar Higgs field $\chi(x)$. The Lagrangian density is given by

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu + igB_\mu)\chi|^2 + \lambda(|\chi|^2 - v^2)^2 \ , \quad (1)$$

where

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \frac{e}{2}\, {}^*\Sigma_{\mu\nu} \quad (2)$$

is the dual field strength tensor. The model is invariant under the U(1) dual gauge transformation, $B_\mu \to B_\mu - \partial_\mu \xi$, $\chi \to \chi e^{i\xi}$, and $\chi^* \to \chi^* e^{-i\xi}$. The strength of the interaction between B_μ and χ is controlled by the dual gauge coupling g and the Higgs self coupling λ, where g satisfies the Dirac quantization condition $(e/2)g = 2\pi$ with the electric charge $e/2$. An important point is that the strong coupling with respect to e is now changed to the weak coupling g. The Higgs condensate v is related to the physical scale.

The structure of the DAH model is quite the same as the Ginzburg-Landau (GL) model for describing various properties of superconductor once B_μ and χ are regarded as the ordinary U(1) gauge field for the electromagnetism and the Cooper pair field, respectively. Thus, the DAH model possesses a string-like flux-tube solution corresponding to the Abrikosov vortex solution in the GL model [4]. Only the difference is the contribution of ${}^*\Sigma_{\mu\nu}$ in the dual field strength tensor in Eq. (2), which is the electric Dirac string and its end points specify the location of the electric charges, namely of the quarks [2].

In the present report, we discuss anatomy of the flux-tube solution in two dimensions by assuming translational invariance along the flux-tube axis. In this case, the system becomes cylindrically symmetric, and finding the solution reduces to a simple one-dimensional problem. In order to investigate the general properties of the flux-tube solution, we need to solve the DAH model numerically in three dimensions as partly performed in [5]. The numerical method demonstrated in the present work will be helpful for further quantitative study in the three-dimensional system.

II. THE FLUX-TUBE SOLUTION (ANALYTICAL STUDY)

For the numerical method demonstrated in the next section, it is useful to isolate the physical scale from the model (the physical scale is taken into account after the solution is obtained anyway). In practice, we may introduce dimensionless variables with carets by

$$gB_\mu \equiv v\hat{B}_\mu \ , \quad \chi \equiv v\hat{\chi} = v(\hat{\phi}_1 + i\hat{\phi}_2) \ ,$$
$$\partial_\mu \equiv v\hat{\partial}_\mu \ , \quad x_\mu \equiv v^{-1}\hat{x}_\mu \ , \quad \Sigma_{\mu\nu} \equiv v^2\hat{\Sigma}_{\mu\nu} \ . \quad (3)$$

Then the action of the DAH model is written as

$$S = \beta v^2 \int d^2\hat{x} \left[\frac{1}{4}\hat{F}_{\mu\nu}^2 + \frac{\hat{m}_B^2}{2}(\hat{D}_\mu\hat{\phi}_a)^2 + \frac{\hat{m}_B^2\hat{m}_\chi^2}{8}(\hat{\phi}_a^2 - 1)^2\right], \quad (4)$$

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where

$$\hat{F}_{\mu\nu} = \hat{\partial}_\mu \hat{B}_\nu - \hat{\partial}_\nu \hat{B}_\mu + 2\pi\, {}^*\hat{\Sigma}_{\mu\nu} \ , \qquad (5)$$

$$\hat{D}_\mu \hat{\phi}_a = \hat{\partial}_\mu \hat{\phi}_a - \epsilon_{ab} \hat{B}_\mu \hat{\phi}_b \ . \qquad (6)$$

The three parameters g, λ, and v in Eq. (1) are now translated to the inverse square of the dual gauge coupling $\beta = 1/g^2$, and the masses of the dual gauge and the Higgs fields,

$$m_B = \sqrt{2} g v \equiv \hat{m}_B v \ , \quad m_\chi = 2\sqrt{\lambda} v \equiv \hat{m}_\chi v \ , \qquad (7)$$

respectively. ϵ_{ab} in the covariant derivative in Eq. (6) is the 2nd-rank antisymmetric tensor ($\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$). The repeated Greek and Latin indices are to be summed over from one to two. For simplicity, we omit all of the caret hereafter.

The DAH model has a nontrivial solution when ${}^*\Sigma_{\mu\nu}(x) \neq 0$ is imposed somewhere, which corresponds to the flux-tube solution. In this case, the dual gauge field is decomposed into two parts, $B_\mu = B_\mu^{\rm reg} + B_\mu^{\rm sing}$, and the second term is determined so as to satisfy the relation [6],

$$\partial_\mu B_\nu^{\rm sing} - \partial_\nu B_\mu^{\rm sing} + 2\pi {}^*\Sigma_{\mu\nu} = 0 \ . \qquad (8)$$

By using the Green function $G(x)$ in two dimensions ($\Delta G(x) = -\delta^{(2)}(x)$ where $G(x-x') = -\frac{1}{2\pi} \ln|x-x'|$), the formal solution of Eq. (8) is expressed as

$$B_\mu^{\rm sing}(x) = -2\pi \sum_{\nu\neq\mu} \int d^2x' G(x-x') \partial'_\nu {}^*\Sigma_{\mu\nu}(x') \ . \qquad (9)$$

For the case that a single Dirac string is put at the origin $x = 0$ such as ${}^*\Sigma_{\mu\nu} = \epsilon_{\mu\nu} N_q \delta^{(2)}(x)$, where N_q characterizes the quark charge, we obtain

$$B_\mu^{\rm sing}(x) = N_q \frac{\epsilon_{\mu\nu} x_\nu}{|x|^2} \ . \qquad (10)$$

In this case, the system becomes cylindrically symmetric. By using the polar coordinate, $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ with $r \equiv |x|$, Eq. (10) becomes

$$\vec{B}^{\rm sing} = -\frac{N_q}{r}(-\sin\theta \vec{e}_1 + \cos\theta \vec{e}_2) = -\frac{N_q}{r}\vec{e}_\theta \ . \qquad (11)$$

Other field variables are also written as

$$\vec{B}^{\rm reg} = B^{\rm reg}(r)\vec{e}_\theta = \frac{\tilde{B}(r)}{r}\vec{e}_\theta \ , \qquad (12)$$

$$\phi_1 = \phi(r)\cos\eta(r) \ , \quad \phi_2 = \phi(r)\sin\eta(r) \ . \qquad (13)$$

We now consider the case that the phase of the Higgs field η is single valued. Then, the phase becomes redundant and can be absorbed into $B^{\rm reg}$, so that the DAH action is written as

$$S = \beta v^2 \int_0^\infty (2\pi r) dr \left[\frac{1}{2}\left(\frac{1}{r}\frac{d\tilde{B}}{dr}\right)^2 + \frac{m_B^2}{2}\left\{ \left(\frac{d\phi}{dr}\right)^2 + \left(\frac{\tilde{B}-N_q}{r}\right)^2 \phi^2 \right\} + \frac{m_B^2 m_\chi^2}{8}(\phi^2-1)^2 \right] . \qquad (14)$$

The field equations are then given by

$$\frac{d^2\tilde{B}}{dr^2} - \frac{1}{r}\frac{d\tilde{B}}{dr} - m_B^2(\tilde{B}-N_q)\phi^2 = 0 \ , \qquad (15)$$

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \left(\frac{\tilde{B}-N_q}{r}\right)^2 \phi - \frac{m_\chi^2}{2}\phi(\phi^2-1) = 0 \ . \qquad (16)$$

The exact analytical solution of these field equations is not known. However, in the region of r where $\phi(r) \sim 1$ is satisfied, the first equation is reduced to

$$\frac{d^2K}{d\rho^2} + \frac{1}{\rho}\frac{dK}{d\rho} - (1 + \frac{1}{\rho^2})K = 0 \ , \qquad (17)$$

where the radial coordinate is rescaled by $\rho = m_B r$, and the dual gauge field is expressed as $\tilde{B} = N_q - c\rho K(\rho)$ with a multiplicative factor c. In fact, the solution of Eq. (17) is known to be the modified Bessel function of the second kind, $K = K_1(\rho) \sim \sqrt{\pi/(2\rho)}e^{-\rho}$ for larger ρ. The electric field and the supercurrent are then expressed as

$$E(r) = \sqrt{\beta}\frac{1}{r}\frac{d\tilde{B}}{dr} = c\sqrt{\beta} m_B^2 K_0(m_B r) \ , \qquad (18)$$

$$k(r) = -\sqrt{\beta} m_B^2 \frac{\tilde{B}-N_q}{r}\phi^2 = c\sqrt{\beta} m_B^3 K_1(m_B r) \ , \qquad (19)$$

indicating the flux-tube structure. Note that the ratio $k(r)/E(r)$ is independent of c, which is expected to approach m_B for larger r, since the asymptotic behavior of K_0 is the same as K_1.

Following the idea of Bogomol'nyi [7], we may rewrite the DAH action in Eq. (14) as

$$S = \beta v^2 \int_0^\infty (2\pi r) dr \left[\frac{1}{2}\left(\frac{1}{r}\frac{d\tilde{B}}{dr} + \frac{m_B^2}{2}(\phi^2-1) \right)^2 + \frac{m_B^2}{2}\left(\frac{d\phi}{dr} + \left(\frac{\tilde{B}-N_q}{r}\right)\phi \right)^2 + \frac{m_B^2(m_\chi^2 - m_B^2)}{8}(\phi^2-1)^2 \right] - \beta v^2 \int_0^\infty (2\pi r) dr \frac{m_B^2}{2r}\left[\frac{d\tilde{B}}{dr}(\phi^2-1) + \frac{d\phi^2}{dr}(\tilde{B}-N_q) \right] . \qquad (20)$$

It is remarkable that in the special case such as $\kappa \equiv m_\chi/m_B = 1$, which is called the Bogomol'nyi limit, the last term of the second line is automatically dropped, and the last line of the action is evaluated analytically only by taking into account the boundary conditions $\tilde{B}(0) = 0$, $\tilde{B}(\infty) = N_q$, and $\phi(0) = 0$, $\phi(\infty) = 1$. As a result, the action simply becomes

$$S = \beta v^2 \cdot N_q \pi m_B^2 \ , \tag{21}$$

where the field equations are given by the first-order differential equations,

$$\frac{1}{r}\frac{d\tilde{B}}{dr} + \frac{m_B^2}{2}(\phi^2 - 1) = 0 \ , \quad \frac{d\phi}{dr} + \left(\frac{\tilde{B} - N_q}{r}\right)\phi = 0 \ . \tag{22}$$

Note that applying the derivative with respect to r recovers the second-order differential equations as in Eqs. (15) and (16).

III. THE FLUX-TUBE SOLUTION (NUMERICAL STUDY)

The existence of the flux-tube solution is known from the analytical investigation as in the previous section. However, in order to clarify the detailed structure of the flux tube, numerical investigation is needed. We may then write the radial coordinate as $r = ns$ $(n = 0, 1, 2, ..., n_{\max})$ with a small interval s and replace the derivative by the (central) difference such that for a function $f(r)$,

$$\frac{df(r)}{dr} \to \frac{f(n+1) - f(n-1)}{2s} \ . \tag{23}$$

Then the DAH action in Eq. (14) is expressed as

$$S = \beta v^2 \sum_n (2\pi n s) s \left[\frac{1}{2}\left(\frac{\tilde{B}(n+1) - \tilde{B}(n-1)}{2ns^2}\right)^2 + \frac{m_B^2}{2}\left\{\left(\frac{\phi(n+1) - \phi(n-1)}{2s}\right)^2 + \left(\frac{\tilde{B}(n) - N_q}{ns}\right)^2 \phi(n)^2\right\} + \frac{m_B^2 m_\chi^2}{8}(\phi(n)^2 - 1)^2 \right] , \tag{24}$$

and the field equations are

$$X(n) \equiv \frac{\tilde{B}(n+1) + \tilde{B}(n-1) - 2\tilde{B}(n)}{s^2} - \frac{\tilde{B}(n+1) - \tilde{B}(n-1)}{2ns^2} - m_B^2(\tilde{B}(n) - N_q)\phi(n)^2 = 0 \ , \tag{25}$$

$$Y(n) \equiv \frac{\phi(n+1) + \phi(n-1) - 2\phi(n)}{s^2} + \frac{\phi(n+1) - \phi(n-1)}{2ns^2} - \left(\frac{\tilde{B}(n) - N_q}{ns}\right)^2 \phi(n) - \frac{m_\chi^2}{2}\phi(n)(\phi(n)^2 - 1) = 0 \ . \tag{26}$$

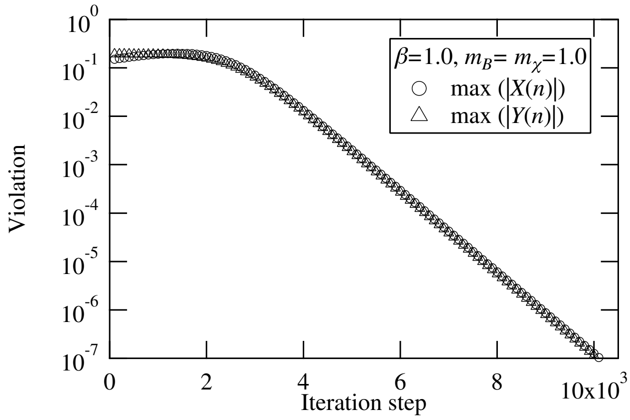

FIG. 1: Histories of the maximum violation of the field equations $\max(|X(n)|)$ and $\max(|Y(n)|)$ as a function of iteration step.

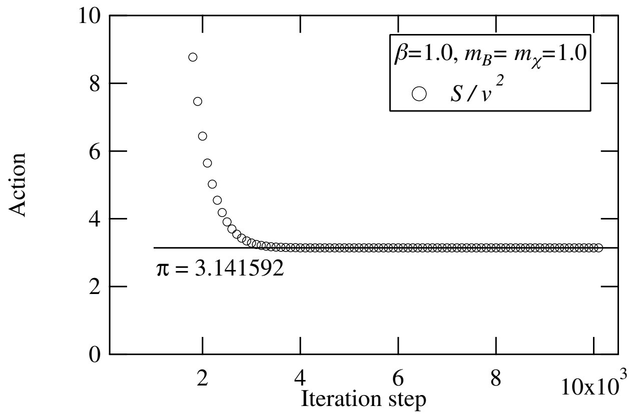

FIG. 2: History of the action S as a function of iteration step, which approaches the analytical expectation $S/v^2 = \pi$ for $\beta = 1$ and $m_B = m_\chi = 1$ with $N_q = 1$.

In order to obtain the solution, we find that the Newton-Raphson method is applicable; the field variables are iteratively updated by

$$\tilde{B}(n) \to \tilde{B}(n) - \frac{X(n)}{X'(n)} \ , \quad \phi(n) \to \phi(n) - \frac{Y(n)}{Y'(n)} \ , \tag{27}$$

where

$$X' = \frac{dX}{d\tilde{B}} = -\frac{2}{s^2} - m_B^2 \phi(n)^2 \ , \tag{28}$$

$$Y' = \frac{dY}{d\phi} = -\frac{2}{s^2} - \left(\frac{\tilde{B}(n) - N_q}{ns}\right)^2 - \frac{m_\chi^2}{2}(3\phi(n)^2 - 1) \ , \tag{29}$$

and the iteration is terminated when $\max(|X(n)|) < \epsilon$ and $\max(|Y(n)|) < \epsilon$ are satisfied simultaneously with a reasonably small value of ϵ. The initial conditions of the dual gauge field and the Higgs field are set to be $\tilde{B}(0) = 0$, $\tilde{B}(n_{\max}) = N_q$, and $\phi(0) = 0$, $\phi(n_{\max}) = 1$ with a reasonably large $n_{\max}$. Using the converged

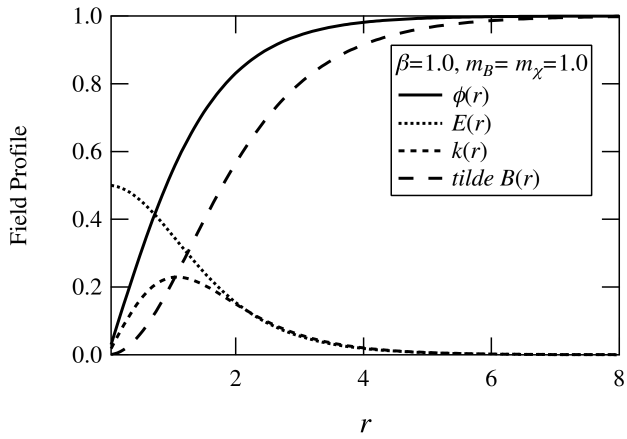

FIG. 3: The field profiles as a function of r for $\beta = 1$ and $m_B = m_\chi = 1$ with $N_q = 1$.

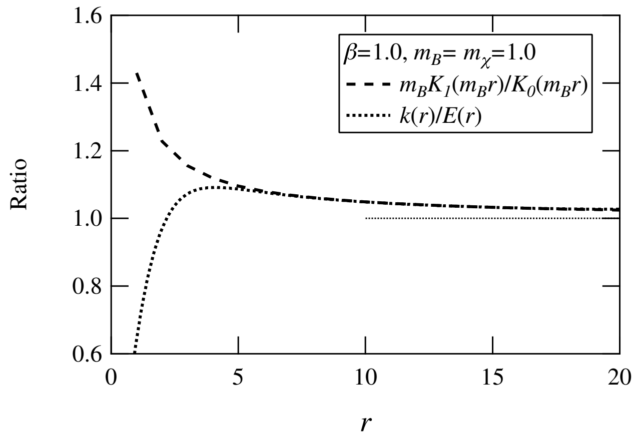

FIG. 4: The ratio of the supercurrent to the electric field as a function of r for $\beta = 1$ and $m_B = m_\chi = 1$ with $N_q = 1$.

values of field variables $\tilde{B}$ and ϕ, the electric field and the supercurrent can be computed as

$$E(r) \;\to\; \sqrt{\beta}\frac{\tilde{B}(n+1) - \tilde{B}(n-1)}{2ns^2}\ , \tag{30}$$

$$k(r) \;\to\; -\sqrt{\beta} m_B^2 \frac{\tilde{B}(n) - N_q}{ns}\phi(n)^2\ . \tag{31}$$

As an example, we demonstrate the case that $\beta = 1.0$ and $m_B = m_\chi = 1.0$ with $N_q = 1$. We show histories of the maximum violation $\max(|X(n)|)$ and $\max(|Y(n)|)$ in Fig. 1, and of the action S in Fig. 2, both as a function of iteration step. We have set $s = 0.05$, $n_{\max} = 800$ ($r_{\max} = 40$), and $\epsilon = 10^{-7}$. We find that the convergence criterion is satisfied after 10^4 steps. It took 1.78 seconds with a single CPU on the Mac mini 3 GHz Intel Core i7 without any specific optimization of the code. The numerical value of the action certainly reproduces the analytical value $S/v^2 = \pi$ already at around 4000 steps. The computation time is of course dependent on the choice of numerical parameters, but the quantitative results can be obtained within affordable time in any case.

We finally plot the field profiles of the Higgs field ϕ, the electric field $E(r)$, the supercurrent $k(r)$, and the modified dual gauge field $\tilde{B}(r)$ in Fig. 3 as a function of r. It now reveals the detailed structure of the flux-tube profile for the whole region of r. We find that the Higgs field is saturated as $\phi \sim 1$ for $r > 5$, where the ratio of the supercurrent to the electric field is well described by that of the modified Bessel functions as shown in Fig. 4, which then approaches $m_B = 1$ for larger r due to $K_0 \sim K_1$.

IV. SUMMARY

We have discussed anatomy of the flux-tube solution in the U(1) DAH model both from analytical and numerical sides. We have found that the numerical method is well controlled, which allows us to further quantitative studies. This method is also applicable to performing a systematic check of numerical investigation of the DAH model in three dimensions [5].

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